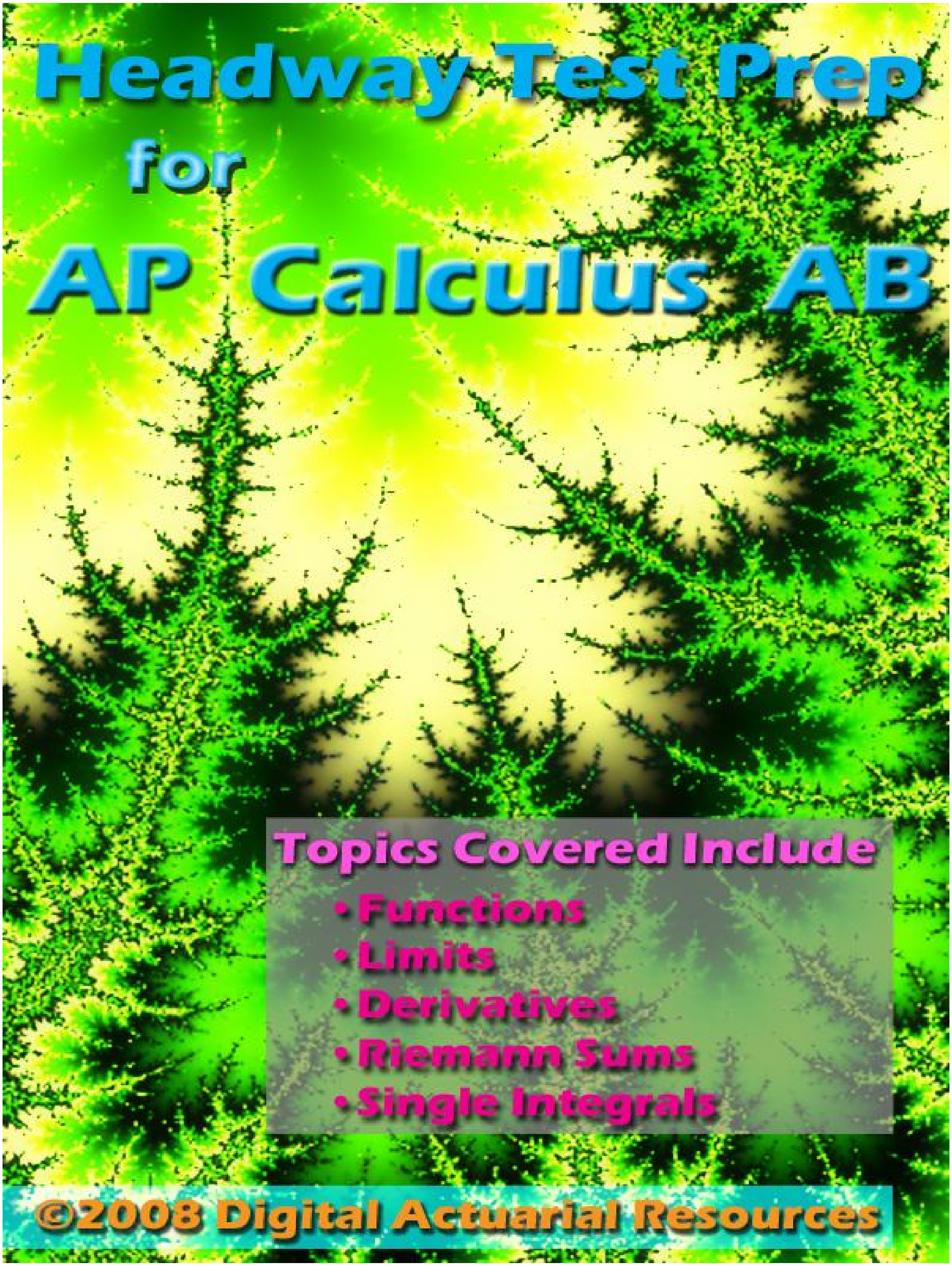
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for
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Section 4-5 Integration by Parts

Integration by parts is helpful when the function inside the integral is the product of two expressions. The contents of the original function are replaced with the symbols 'u' and "dv," which are multiplied together. We also need to find the derivative of u, called "du," and the anti-derivative of dv, which is 'v.' Integration by parts entails reversing the product rule for derivatives. Recall that the product rule is:

$$(uv)' = u'v + uv'$$

Integrating the product rule with indefinite integrals yields:

$$\int (uv)' = \int u'v + uv' = \int u'v + \int uv'$$

Consider the very first integral on the left side. Remember that the integral of a derivative is the inner function (the function being differentiated). Therefore, we can set the above integrals equal to uv:

$$\int u'v + \int uv' = uv$$

Rearranging produces:

$$\int uv' = uv - \int u'v$$

The integration by parts formulas are slightly different for definite and indefinite integrals:

Integration by Parts Formula for Indefinite Integrals

$$\int u * dv = uv - \int du * v$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u * dv = [uv]_a^b - \int_a^b du * v$$

As an example, consider the definite integral below:

$$\int_0^1 15x * e^{2x} \quad dx$$

To evaluate the integral by conventional means, we need to find the anti-derivative for the function in the integral. What function, when differentiated, would produce $15x * e^{2x}$? Finding this anti-derivative is most difficult. We could, however, break the function into two pieces. One part will be labeled 'u,' and the other component is "dv." Suppose we choose u and dv as follows:

let $u = 15x$
let $dv = e^{2x} \quad dx$

The expression for dv will always contain "dx," where x is the initial independent variable. The formula for by-parts integration becomes:

$$\int_0^1 15x * e^{2x} \quad dx = [uv]_0^1 - \int_0^1 du * v$$

To complete the formula, we need expressions for du and v.

$$du = 15 \quad dx$$
$$v = \frac{1}{2} e^{2x}$$

du is the derivative of u, and v is the anti-derivative of dv.

$$\int_0^1 15x * e^{2x} \quad dx = \left[15x * \frac{1}{2} e^{2x}\right]_0^1 - \int_0^1 15 * \frac{1}{2} e^{2x} \quad dx$$

Calculating the anti-derivative and the integral separately…

$$\left[15x * \frac{1}{2} e^{2x}\right]_0^1 = 7.5(1 * e^{2*1} - 0 * e^{2*0}) = 7.5e^2$$

$$\int_0^1 15 * \frac{1}{2} e^{2x} \quad dx = 7.5\int_0^1 e^{2x} \quad dx = 7.5 * \left[\frac{1}{2} e^{2x}\right]_0^1 = 3.75 * (e^{2*1} - e^{2*0}) = 3.75e^2 - 3.75$$

$$\int_0^1 15x * e^{2x} \quad dx = 7.5e^2 - (3.75e^2 - 3.75) = 3.75e^2 + 3.75$$

The next example shows the high importance of choosing the correct expression for u.

Example
Calculate the following integral using integration by parts:

$$\int_4^7 x * \sin(x)\ dx$$

Solution:
(a.)
The first step in integrating by parts is to choose the expressions for u and dv. Suppose we let $u = \sin(x)$ and $dv = x\ dx$. Now,

let $u = \sin(x)$ $\rightarrow$ $du = \cos(x)\ dx$
let $dv = x\ dx$ $\rightarrow$ $v = \frac{1}{2}x^2$

Note that the "dx" symbol must be included in the expressions for du and dv.

$$\int_4^7 x * \sin(x)\ dx = \left[\sin(x) * \frac{1}{2}x^2\right]_4^7 - \int_4^7 \cos(x) * \frac{1}{2}x^2\ dx$$

The first anti-derivative can be evaluated by placing 7 and 4 in the expression. The integral on the right side will require another round of integration by parts. Let's set $u = \cos(x)$ and $dv = \frac{1}{2}x^2\ dx$. Now,

let $u = \cos(x)$ $\rightarrow$ $du = -\sin(x)\ dx$
let $dv = \frac{1}{2}x^2\ dx$ $\rightarrow$ $v = \frac{1}{6}x^3$

$$\int_4^7 \cos(x) * \frac{1}{2}x^2\ dx = \left[\cos(x) * \frac{1}{6}x^3\right]_4^7 - \int_4^7 -\sin(x) * \frac{1}{6}x^3\ dx$$

This integral has become increasingly more complicated. We can evaluate the anti-derivative on the left side, but the integral on the right side would require another round of integration by parts. This cycle of integration by parts appears to be continuing indefinitely. Did we choose the wrong value for 'u' at the start?

(b.)
Suppose we let $u = x$ and $dv = \sin(x)\ dx$. The integration by parts process becomes:

let $u = x$ $\rightarrow$ $du = 1\ dx$
let $dv = \sin(x)\ dx$ $\rightarrow$ $v = -\cos(x)$

$$\int_4^7 x * \sin(x)\ dx = [-x * \cos(x)]_4^7 - \int_4^7 -\cos(x)\ dx$$

$$= \left((-7 * \cos(7)) - (-4 * \cos(4))\right) + \int_4^7 \cos(x) \; dx$$
$$= (-5.27729) - (2.61456) + [\sin(x)]_4^7$$
$$= -7.89185 + (\sin(7) - \sin(4))$$
$$= -7.89185 + 0.65699 - (-0.7568)$$
$$= -6.47626$$

If you become stuck in a repetitive cycle of integration by parts, start the problem over with a different value of 'u.'

Use the key below when choosing the expression for u. The first choice for u should be logarithmic functions, such as natural logs. If the original function does not contain logs, then search for inverse trigonometric functions. Work down the list until you find an expression that exists within your original integral, and assign it to u. If you choose an expression for dv that is higher on the list than the expression type for u, you might become caught in the integration by parts loop. You should try to find a value for u that is easy to differentiate, and a value for dv that is easy to anti-differentiate. The new integral that results from by-parts holds $du * v$, which must also be easy to integrate.

Optimal Choice for u:
L -- Logs
I -- Inverse Trig Functions
A -- Algebraic Functions
T -- Trig Functions
E -- Exponential Functions

Example
Solve the following integral:

$$\int_0^{10} 4e^x * x^2 \; dx$$

Solution:
Using the LIATE key, u must be $4x^2$, the algebraic expression.

let $u = 4x^2 \quad \rightarrow \quad du = 8x \; dx$
let $dv = e^x \; dx \quad \rightarrow \quad v = e^x$

$$\int_0^{10} 4e^x * x^2 \; dx = \left[4x^2 * e^x\right]_0^{10} - \int_0^{10} 8xe^x \; dx$$

$$\left[4x^2 * e^x\right]_0^{10} = 4*10^2 * e^{10} - 4*0^2 * e^0 = 400e^{10}$$

Solving $\int_0^{10} 8xe^x \ dx$ will require more integration by parts.

let $u = 8x \quad \rightarrow \quad du = 8 \ dx$

let $dv = e^x \ dx \quad \rightarrow \quad v = e^x$

$$\int_0^{10} 8xe^x \ dx = \left[8x * e^x\right]_0^{10} - \int_0^{10} 8e^x \ dx$$
$$= (8*10*e^{10} - 8*0*e^0) - \left[8e^x\right]_0^{10}$$
$$= 80e^{10} - (8e^{10} - 8e^0) = 72e^{10} + 8$$

$$\int_0^{10} 4e^x * x^2 \ dx = \left[4x^2 * e^x\right]_0^{10} - \int_0^{10} 8xe^x \ dx = 400e^{10} - (72e^{10} + 8) = 328e^{10} - 8$$

Example
Find the indefinite integral for $f(x) = \ln(x)$.

Solution:
Must Find: $\int \ln(x) \ dx$

let $u = \ln(x) \quad \rightarrow \quad du = \frac{1}{x} \ dx$

let $dv = 1 \ dx \quad \rightarrow \quad v = x$

$$\int \ln(x) \ dx = \ln(x) * x - \int \frac{1}{x} * x \ dx = \ln(x) * x - \int 1 \ dx = \ln(x) * x - (x + c)$$
$$= \ln(x) * x - x + c$$

Example

Evaluate this integral: $\int_0^1 \cos(x) * 8x^2 \ dx$

Solution:
let $u = 8x^2 \quad \rightarrow \quad du = 16x \ dx$

let $dv = \cos(x) \ dx \quad \rightarrow \quad v = \sin(x)$

$$\int = \left[\sin(x)*8x^2\right]_0^1 - \int_0^1 16x*\sin(x)\,dx$$

$$= 6.73177 - \int_0^1 16x*\sin(x)\,dx$$

let $u = 16x$ → $du = 16 \ dx$

let $dv = \sin(x)$ → $v = -\cos(x)$

$$\int_0^1 16x*\sin(x)\,dx = \left[-16x*\cos(x)\right]_0^1 - \int_0^1 -16\cos(x)\,dx$$

$$= -8.64484 + \int_0^1 16\cos(x)\,dx$$

$$= -8.64484 + \left[16\sin(x)\right]_0^1 = -8.64484 + 13.46354 = 4.8187$$

original integral = 6.73177 – 4.8187 = 1.91307

Example

Evaluate this integral: $\int_2^5 \ln(x^2)*(x^7+13x^2)\,dx$

Solution:

let $u = \ln(x^2)$ → $du = \frac{1}{x^2}*2x \ dx = \frac{2}{x}\,dx$

let $dv = x^7 + 13x^2 \ dx$ → $v = \frac{1}{8}x^8 + \frac{13}{3}x^3$

$$\int = \left[\ln(x^2)*\left(\frac{1}{8}x^8+\frac{13}{3}x^3\right)\right]_2^5 - \int_2^5 \frac{2\left(\frac{1}{8}x^8+\frac{13}{3}x^3\right)}{x}\,dx$$

$$= 158{,}823 - \int_2^5 \frac{1}{4}x^7 + \frac{26}{3}x^2 \ dx$$

$$= 158{,}823 - \left[\frac{1}{32}x^8 + \frac{26}{9}x^3\right]_2^5 = 158{,}823 - (12{,}568 - 31) = 146{,}286$$